Informed Search: A* Algorithm

CE417: Introduction to Artificial Intelligence
Sharif University of Technology
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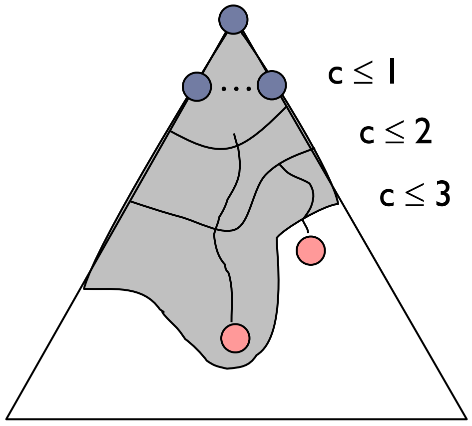
Most slides have been adopted from Klein and Abdeel, CS188, UC Berkeley.

Uninformed search

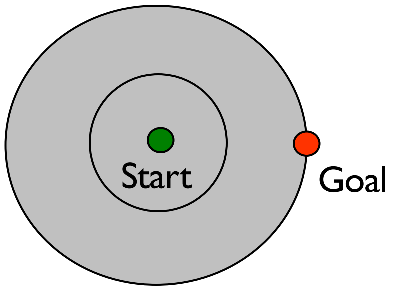
Uniform Cost Search

- Strategy: expand lowest path cost



- The good: UCS is complete and optimal!

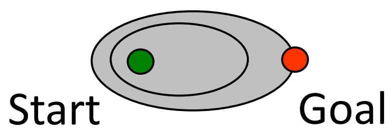
- The bad:
  - Explores options in every "direction"
  - No information about goal location



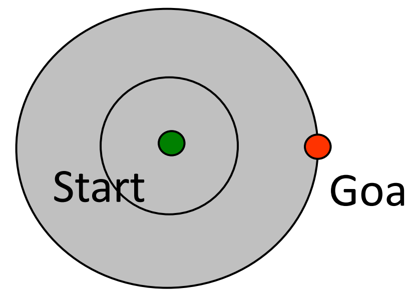
UCS example

What we would like to have happen

Guide search ***towards the goal*** instead of ***all over the place***

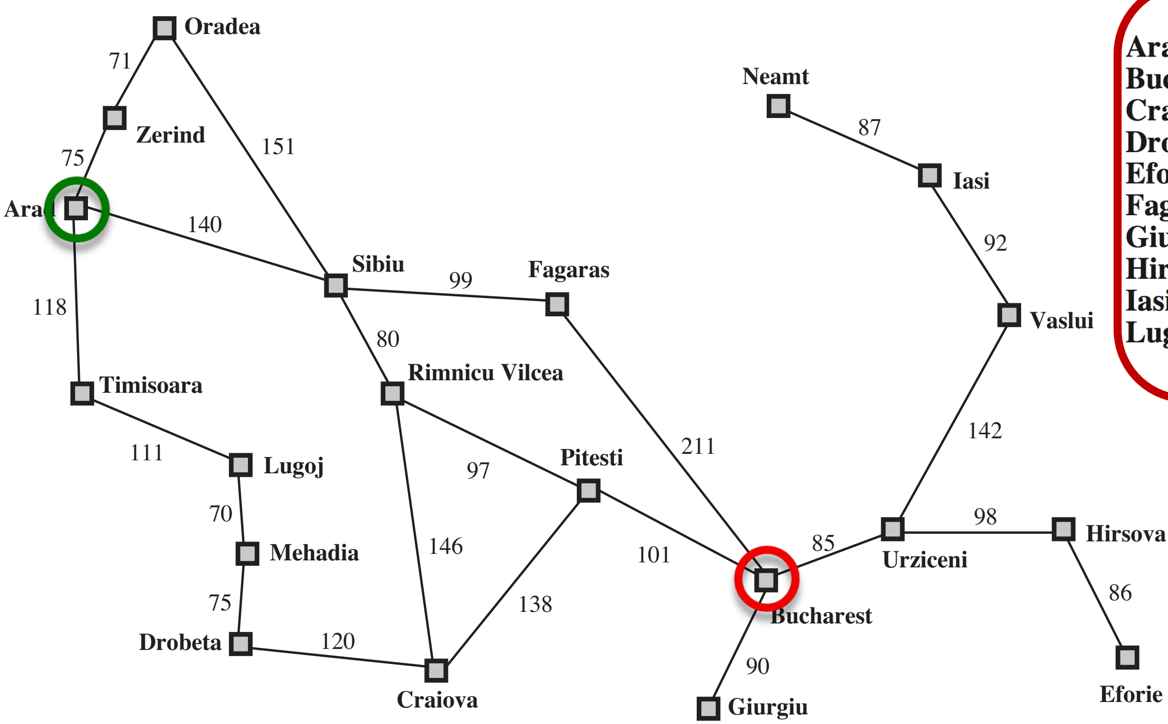


Informed



Uninformed

Example: Route-finding in Romania

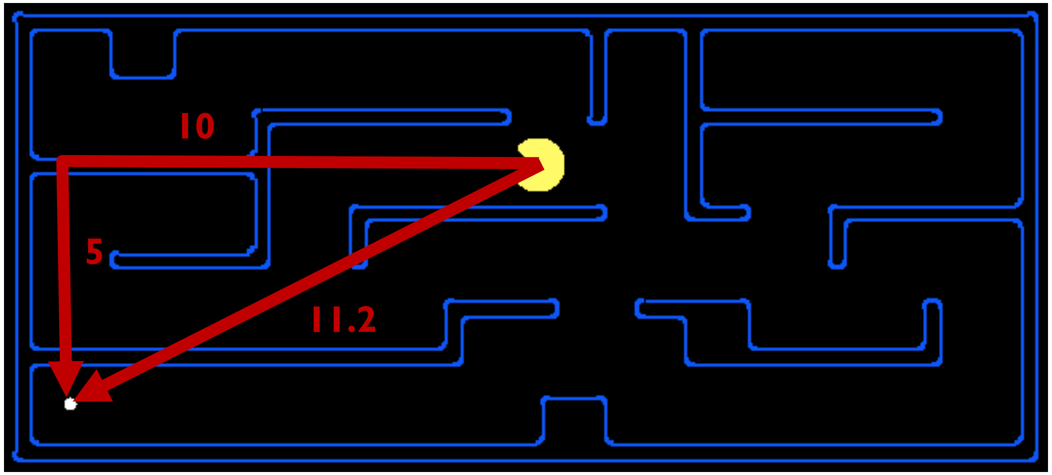


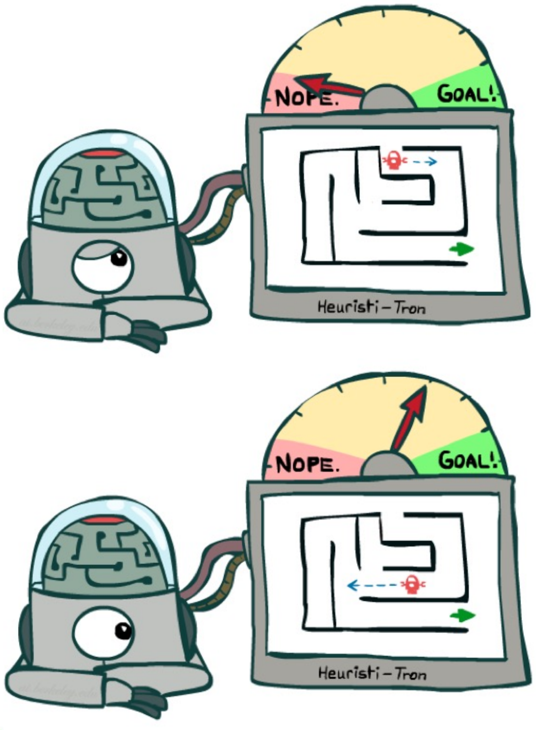
Arad	366	**Mehadia**	241
Bucharest	0	**Neamt**	234
Craiova	160	**Oradea**	380
Drobeta	242	**Pitesti**	100
Eforie	161	**Rimnicu Vilcea**	193
Fagaras	176	**Sibiu**	253
Giurgiu	77	**Timisoara**	329
Hirsova	151	**Urziceni**	80
Iasi	226	**Vaslui**	199
Lugoj	244	**Zerind**	374

$h(n)$ = straight-line distance to Bucharest

Search heuristics

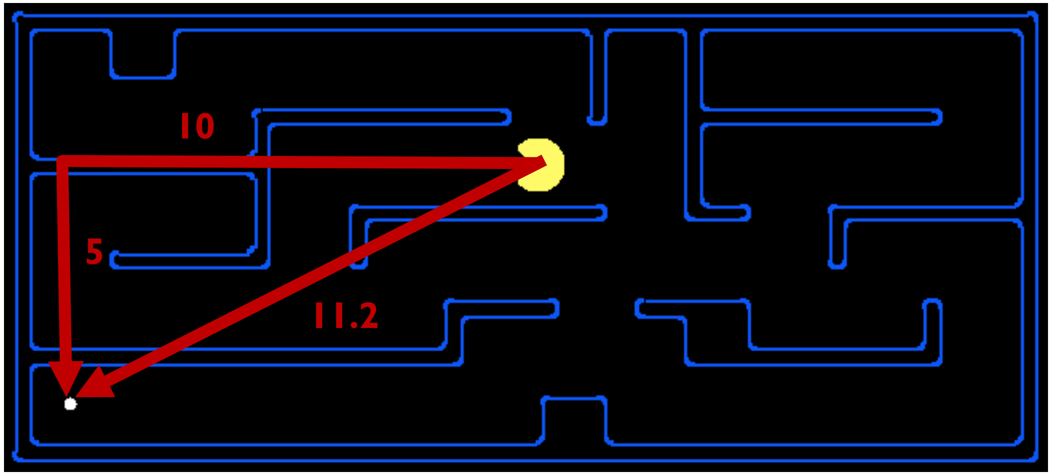
- A heuristic is:
  - A function that *estimates* how close a state is to a goal
  - Designed for a particular search problem
  - Examples: Manhattan distance, Euclidean distance for pathing

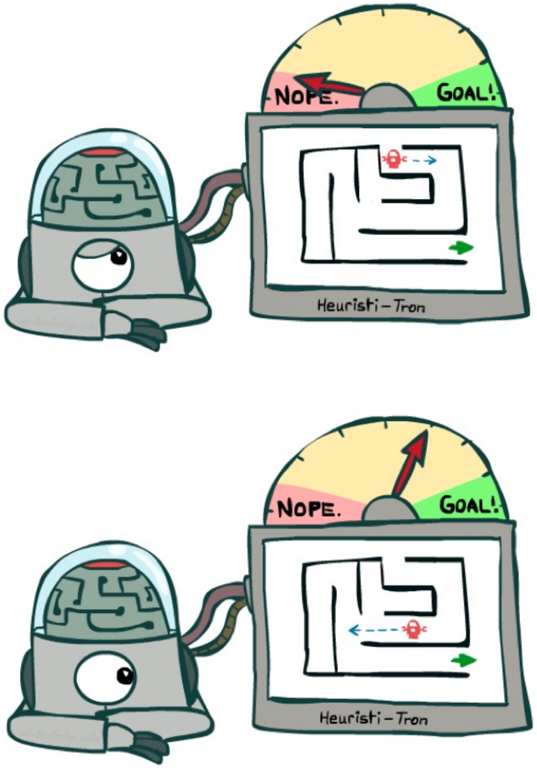




Example: Pathing in pacman

- $h(n)$ = Manhattan distance = $|\Delta x| + |\Delta y|$
- Is Manhattan better than straight-line distance?





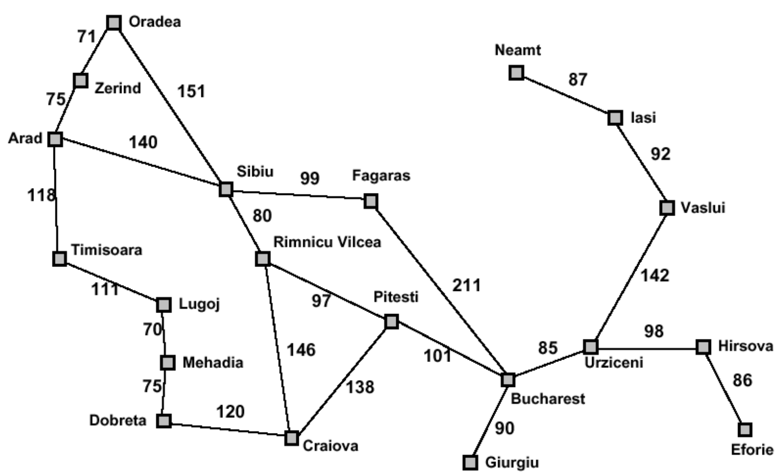
Greedy Best First Search

Greedy Best First Search

- Priority queue based on $h(n)$
  - e.g., $h_{SLD}(n)$ = straight-line distance from n to Bucharest

- Greedy search expands the node that appears to be closest to goal

Greedy



Straight–line distance to Bucharest

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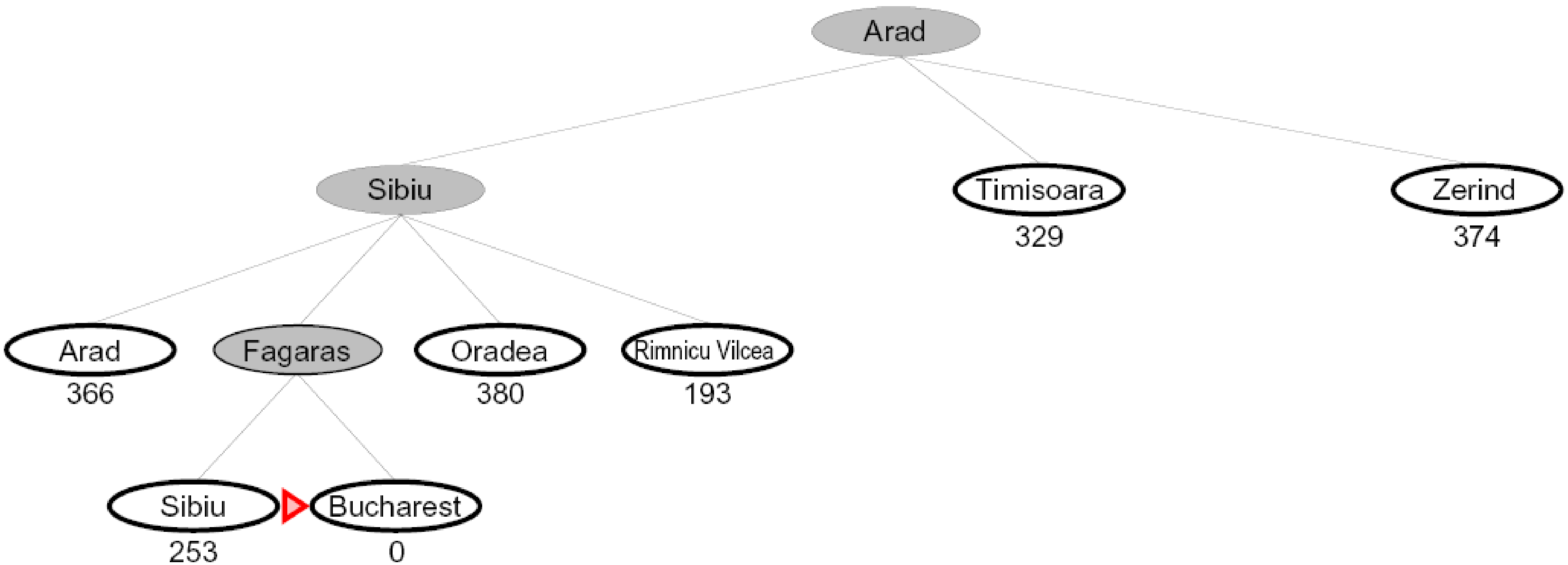
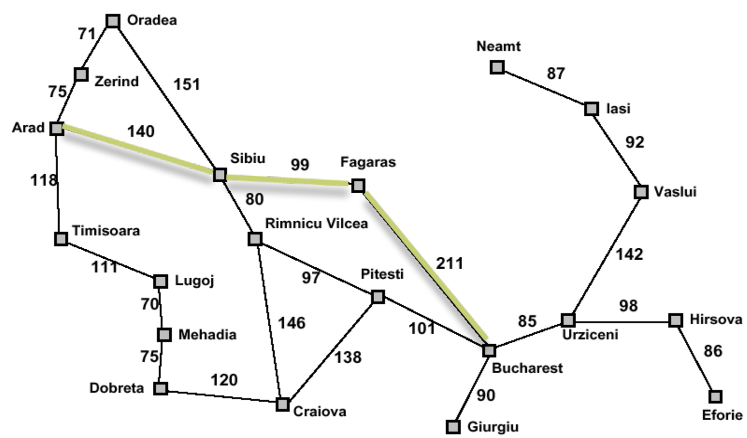
Greedy Best First Search





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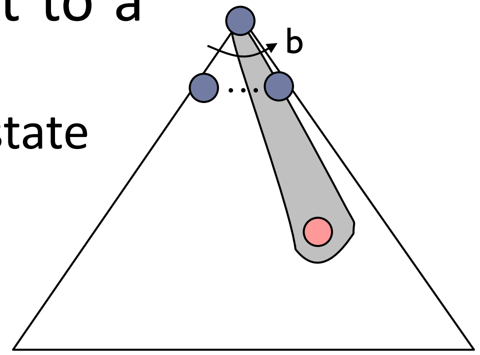
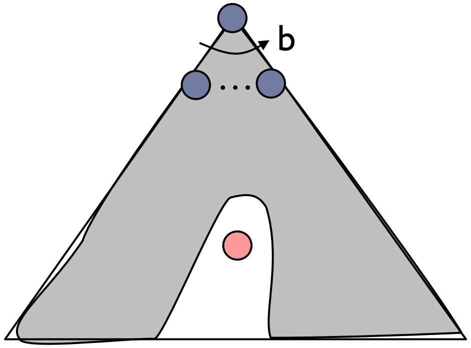
Greedy Best First Search

- Strategy: expand a node that you think is closest to a goal state
  - Heuristic: estimate of distance to nearest goal for each state



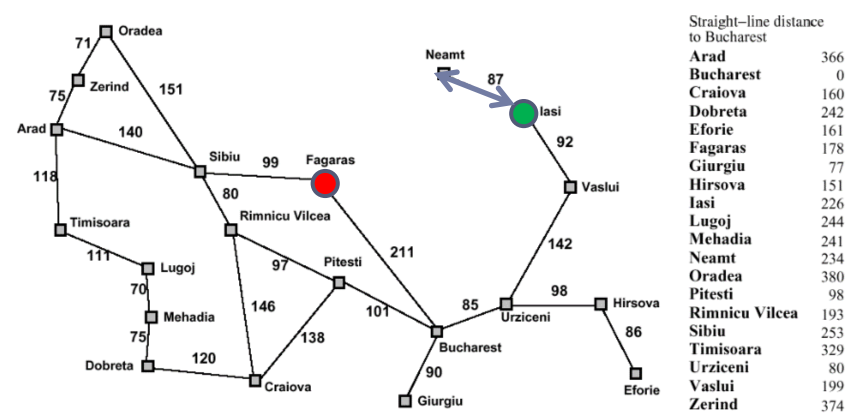
- A common case:
  - Best-first takes you straight to the (wrong) goal



- Worst-case: like a badly-guided DFS

Properties of greedy search

- Complete? No
  - Similar to DFS, only graph search version is complete in finite spaces
  - Infinite loops, e.g., (Iasi to Fagaras) Iasi → Neamt → Iasi → Neamt
- Time
  - $O(b^m)$, but a good heuristic can give dramatic improvement
- Space
  - $O(b^m)$: keeps all nodes in memory
- Optimal? No



Video of demo contours greedy (Empty)

Video of demo contours greedy (Pacman small maze)

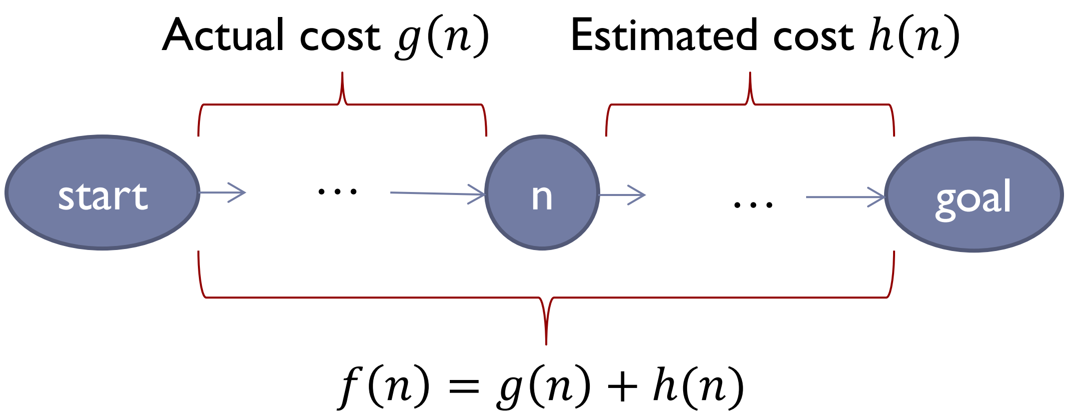
A* search

A^*: The core idea

- Expand a node *n* most likely to be on an optimal path
- Expand a node *n* s.t. the cost of the best solution through *n* is optimal
- Expand a node *n* with lowest value of $g(n) + h^*(n)$
  - $g(n)$ is the cost from root to *n*
  - $h^*(n)$ is the optimal cost from *n* to the closest goal
- We seldom know $h^*(n)$ but might have a heuristic approximation $h(n)$
- A^* = tree search with priority queue ordered by $f(n) = g(n) + h(n)$

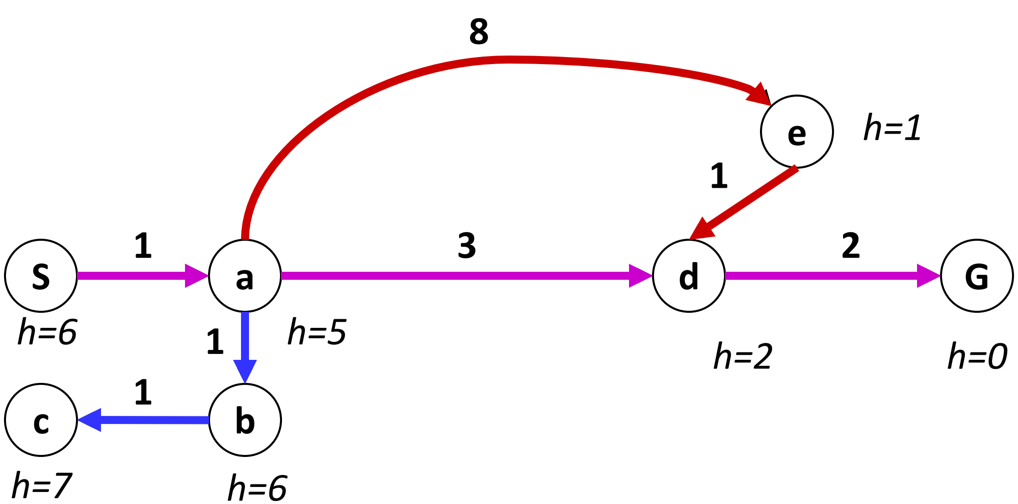
A* search

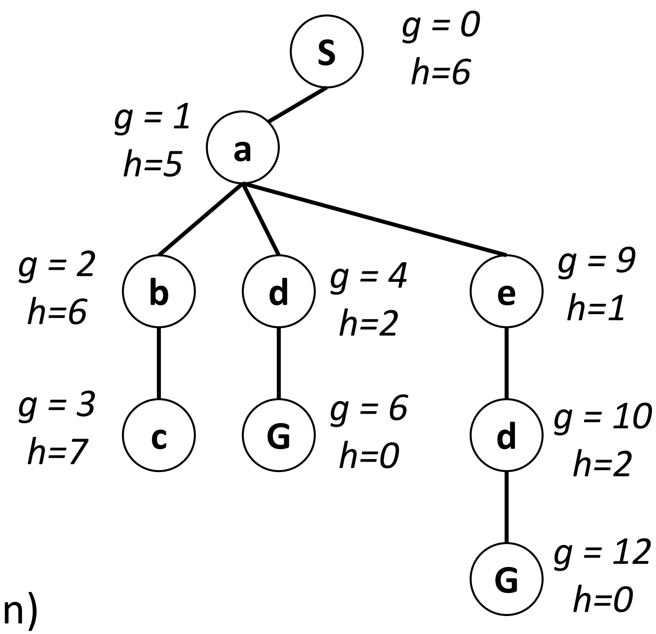
- Idea: minimizing the total estimated solution cost
- Evaluation function for priority $\underline{f(n) = g(n) + h(n)}$
  - $g(n) =$ cost so far to reach n
  - $h(n) =$ estimated cost of the cheapest path from n to goal
  - So, $f(n) =$ estimated total cost of path through n to goal



A*: Combining UCS and Greedy

- Uniform-cost orders by path cost, or *backward cost* g(n)
- Greedy orders by goal proximity, or *forward cost* h(n)



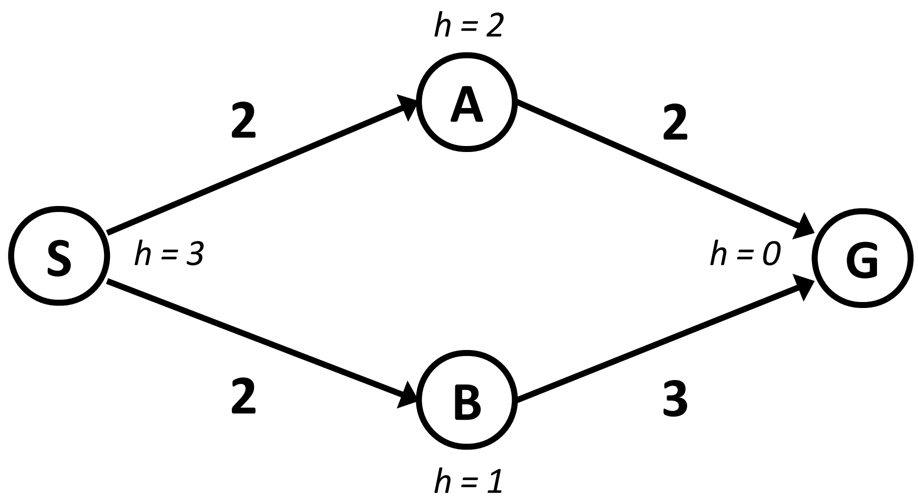


- A* Search orders by the sum: f(n) = g(n) + h(n)

Example: Teg Grenager

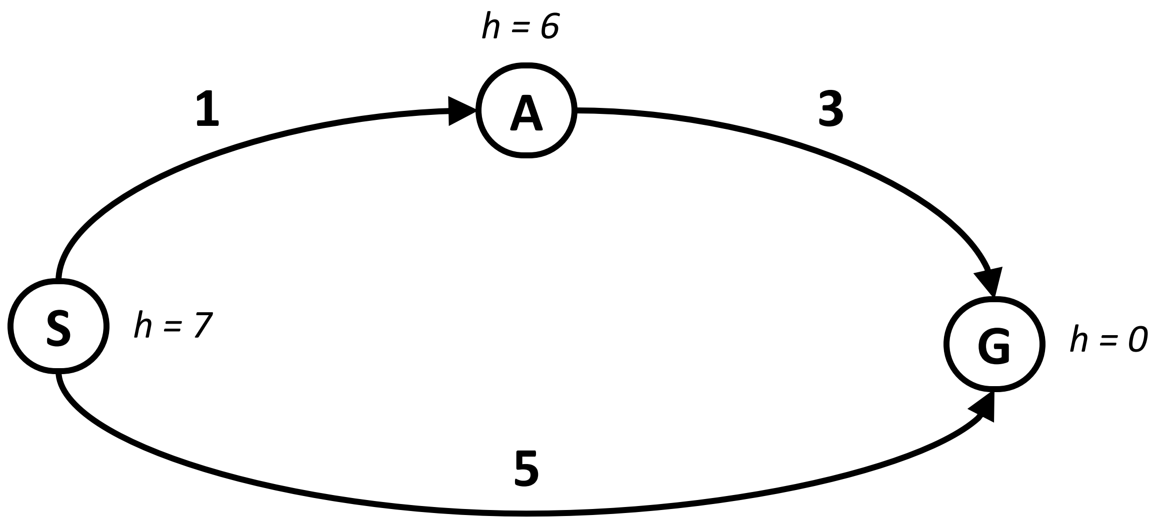
A* termination

- Should we stop when we enqueue a goal?



- No: only stop when we dequeue a goal

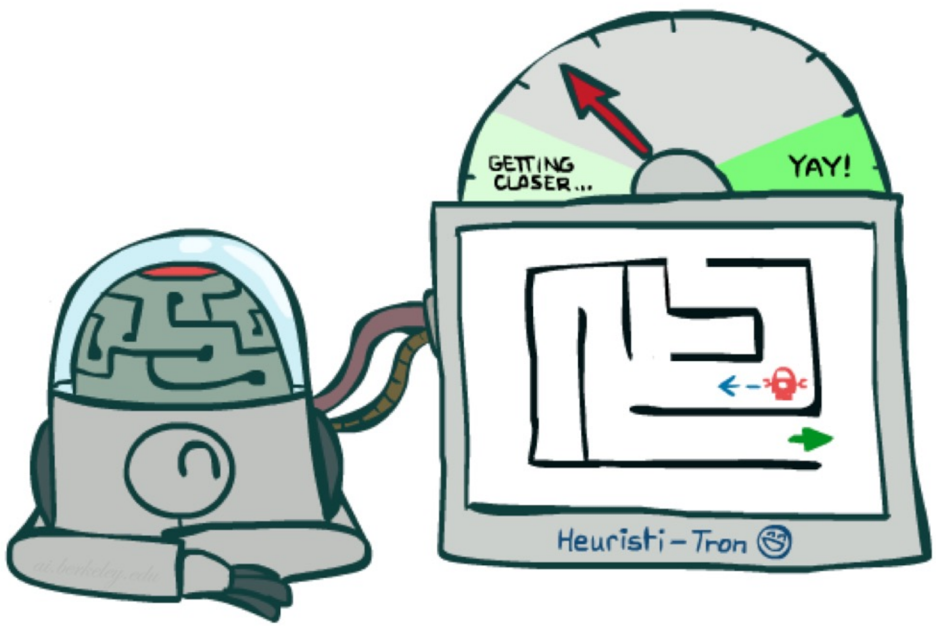
Is A* Optimal?



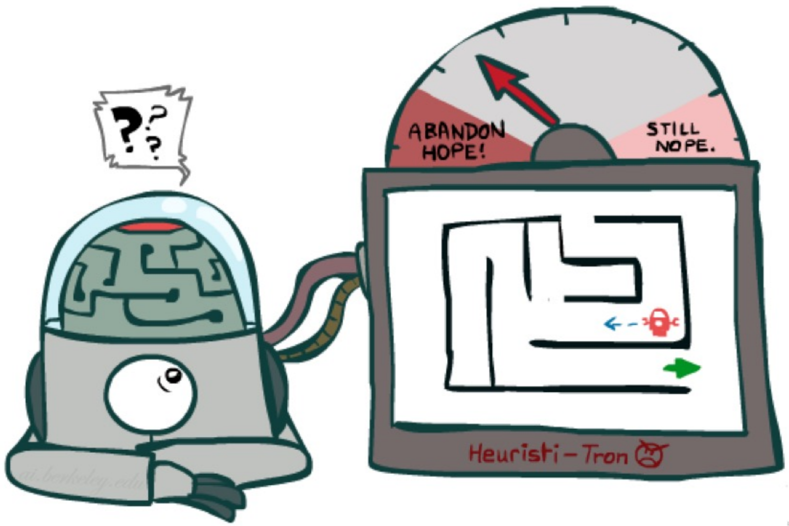
What went wrong?

- ***Actual*** bad solution cost < ***estimated*** good solution cost
- We need estimates to be less than actual costs!

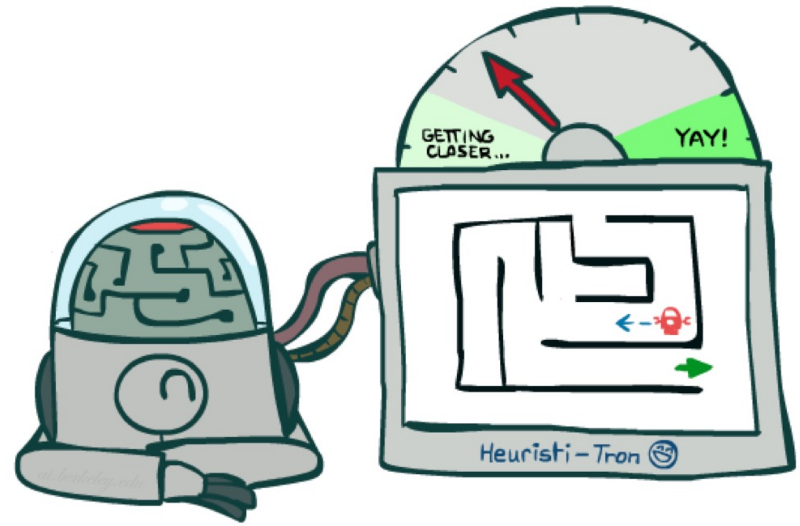
Admissible Heuristics



Idea: Admissibility



Inadmissible (pessimistic) heuristics break optimality by trapping good plans on the frontier



Admissible (optimistic) heuristics slow down bad plans but never outweigh true costs

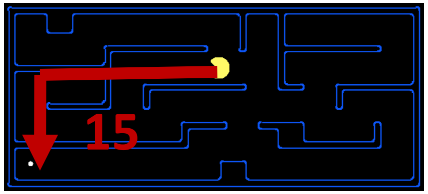
Admissible Heuristics

- A heuristic *h* is ***admissible*** (optimistic) if:

$$0 \leq h(n) \leq h^*(n)$$

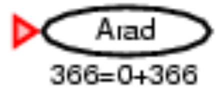
- where $h^*(n)$ is the true cost to a nearest goal

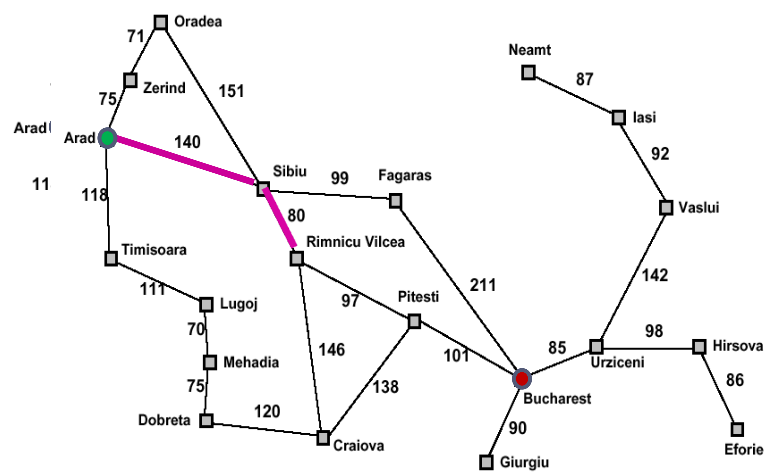
- Examples:



- Coming up with admissible heuristics is most of what's involved in using A* in practice.

A* search: example





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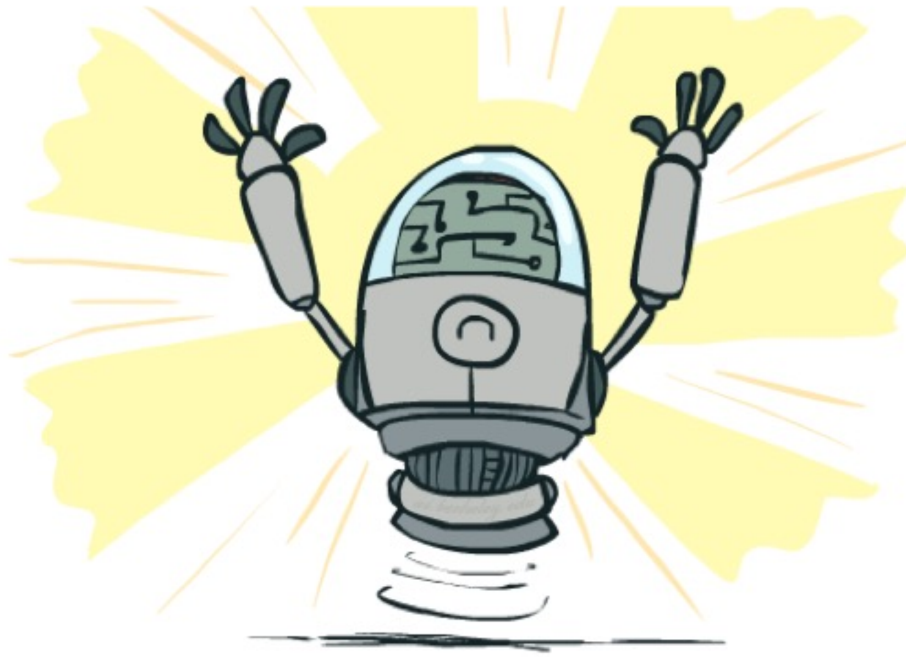
Optimality of A* Tree Search



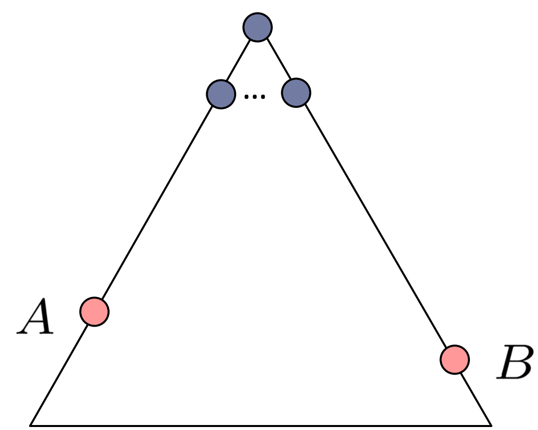
Optimality of A* Tree Search

Assume:

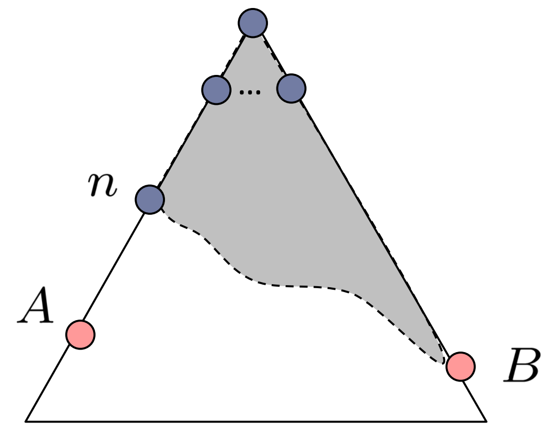
- A is an optimal goal node
- B is a suboptimal goal node
- h is admissible

Claim:

- A will exit the frontier before B



Optimality of A* Tree Search: Blocking



Proof:

- Imagine B is on the frontier
- Some ancestor *n* of A is on the frontier, too (maybe A!)
- Claim: *n* will be expanded before B
  1. f(n) is less or equal to f(A)

Optimality of A* Tree Search: Blocking

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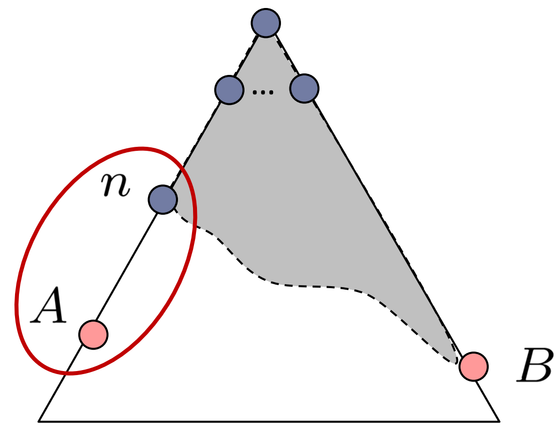
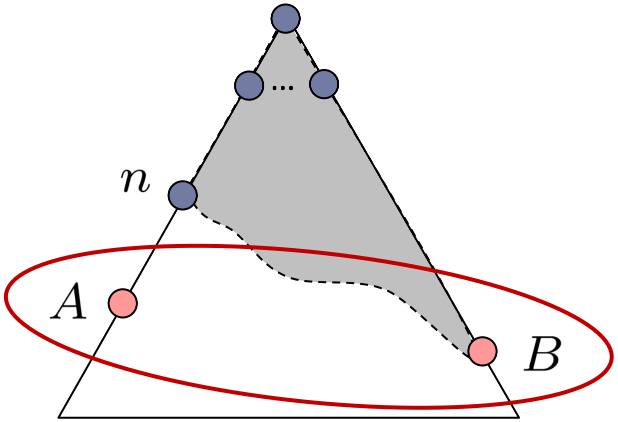


$f(n) = g(n) + h(n)$	Definition of f-cost
$f(n) \leq g(n) + h^*(n)$	Admissibility of h
$f(n) \leq g(A)$	$g(A) = g(n) + h^*(n)$
$g(A) = f(A)$	h = 0 at a goal

Optimality of A* Tree Search: Blocking

Proof:

- Imagine B is on the frontier
- Some ancestor *n* of A is on the frontier, too (maybe A!)
- Claim: *n* will be expanded before B
  1. f(n) is less or equal to f(A)
  2. f(A) is less than f(B)



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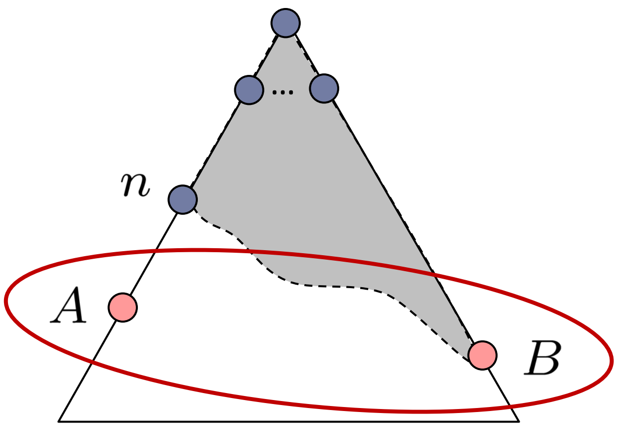


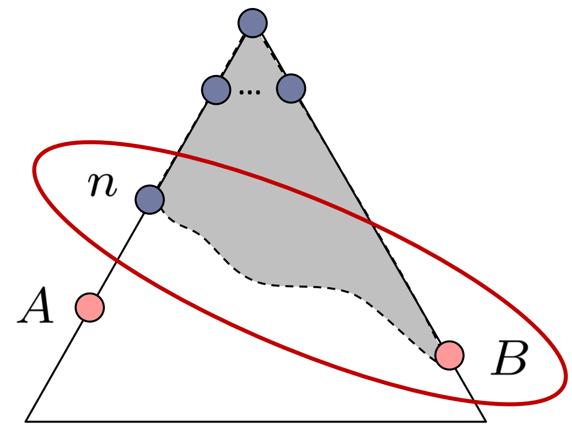
$g(A) < g(B)$ B is suboptimal

$f(A) < f(B)$ h = 0 at a goal

Optimality of A* Tree Search: Blocking

Proof:

- Imagine B is on the frontier
- Some ancestor *n* of A is on the frontier, too (maybe A!)
- Claim: *n* will be expanded before B
  1. f(n) is less or equal to f(A)
  2. f(A) is less than f(B)
  3. *n* expands before B

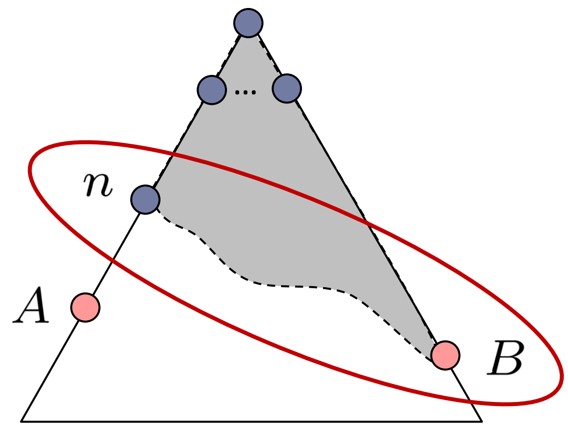


$$f(n) \leq f(A) < f(B)$$

Optimality of A* Tree Search: Blocking

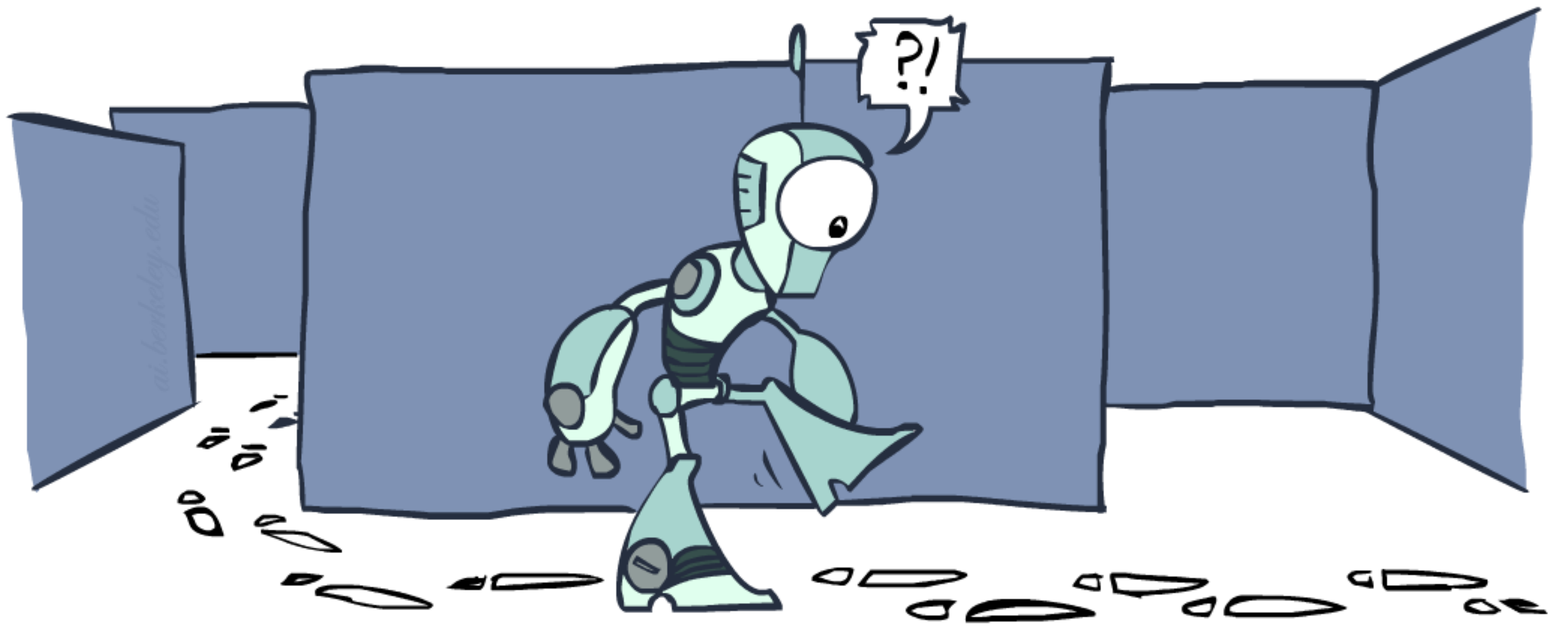
Proof:

- Imagine B is on the frontier
- Some ancestor *n* of A is on the frontier, too (maybe A!)
- Claim: *n* will be expanded before B
  1. f(n) is less or equal to f(A)
  2. f(A) is less than f(B)
  3. *n* expands before B
- All ancestors of A expand before B
- A expands before B
- A* search is optimal



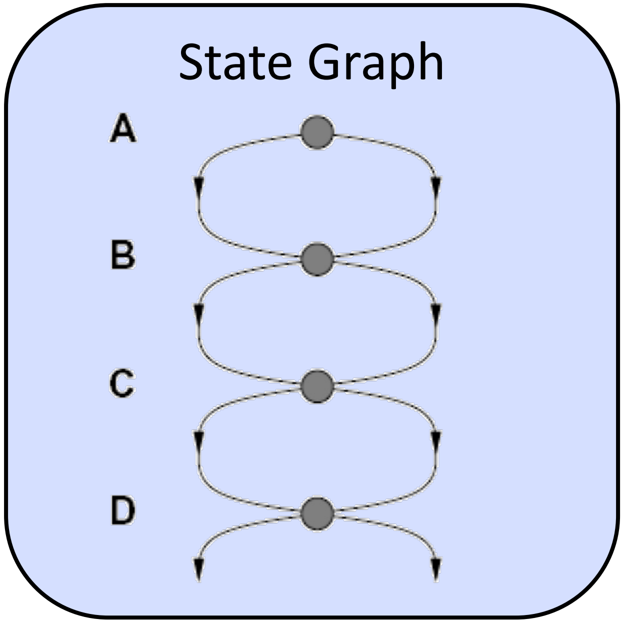
$$f(n) \leq f(A) < f(B)$$

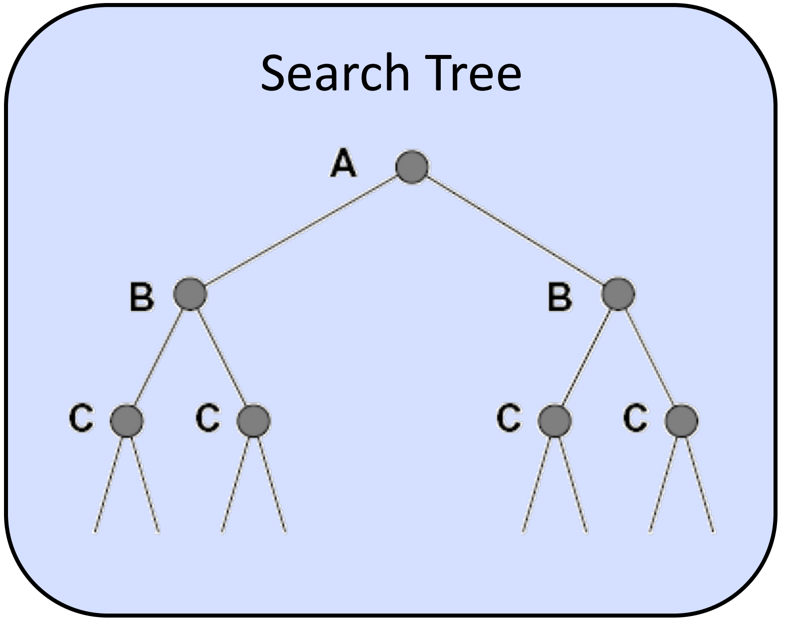
Graph Search



Tree Search: Extra Work!

- Failure to detect repeated states can cause exponentially more work.



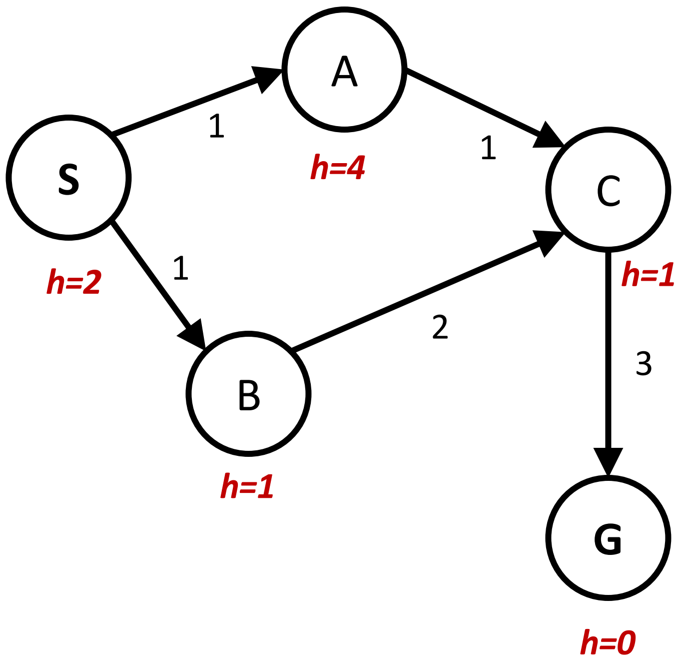


Graph Search

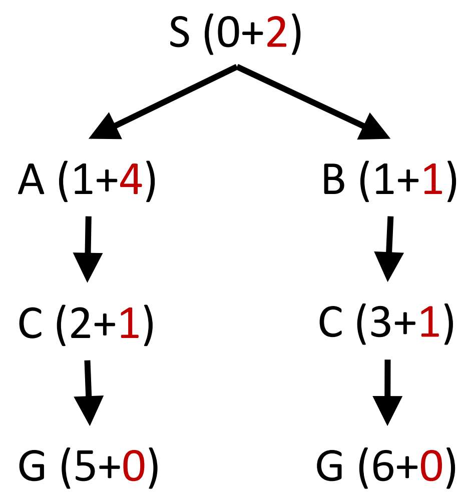
- Idea: never expand a state twice
- How to implement:
  - Tree search + set of expanded states ("closed set")
  - Expand the search tree node-by-node, but…
  - Before expanding a node, check to make sure its state has never been expanded before
  - If not new, skip it, if new add to closed set
- Important: store the closed set as a set, not a list
- Can graph search wreck completeness? Why/why not?
- How about optimality?

A* Graph Search Gone Wrong?

State space graph



Search tree



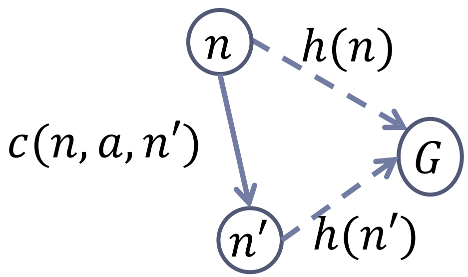
Simple check against expanded set blocks C

Fancy check allows new C if cheaper than old but requires recalculating C's descendants

Conditions for optimality of A*

- Admissibility: $h(n)$ be a lower bound on the cost to reach goal
  - Condition for optimality of TREE−SEARCH version of A*

- Consistency (monotonicity): $h(n) \leq c(n,a,n') + h(n')$
  - Condition for optimality of GRAPH−SEARCH version of A*



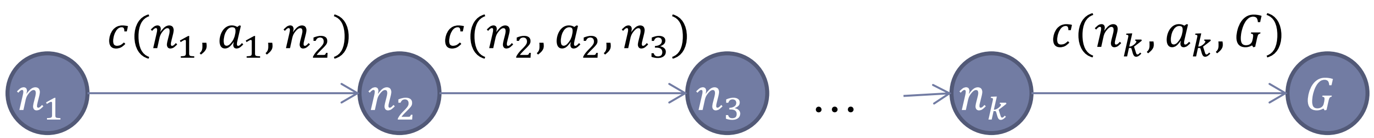
for every node n and every successor n' generated by any action a

$$h(n) \leq c(n,a,n') + h(n')$$

$c(n,a,n')$: cost of generating n' by applying action to n

Consistency implies admissibility

- Consistency ⇒ Admissblity
  - All consistent heuristic functions are admissible
  - Nonetheless, most admissible heuristics are also consistent



$$
\begin{aligned}
h(n_1) &\le c(n_1, a_1, n_2) + h(n_2) \\
&\le c(n_1, a_1, n_2) + c(n_2, a_2, n_3) + h(n_3) \\
&\dots \\
&\le \textstyle\sum_{i=1}^{k} c(n_i, a_i, n_{i+1}) + h(\mathrm{G})
\end{aligned}
$$

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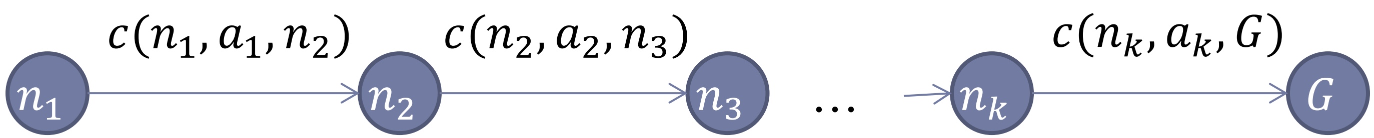

$$
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&\dots \\
&\leq \sum_{i=1}^{k} c(n_i, a_i, n_{i+1}) + \underbrace{h(G)}_{0}
\end{aligned}
$$

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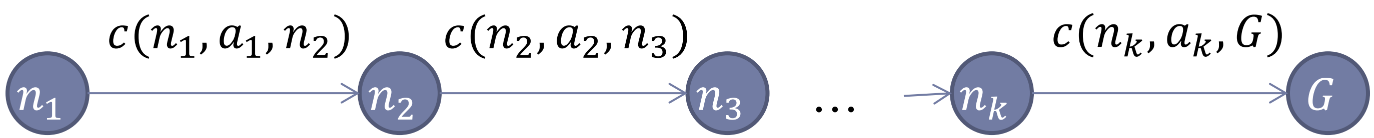

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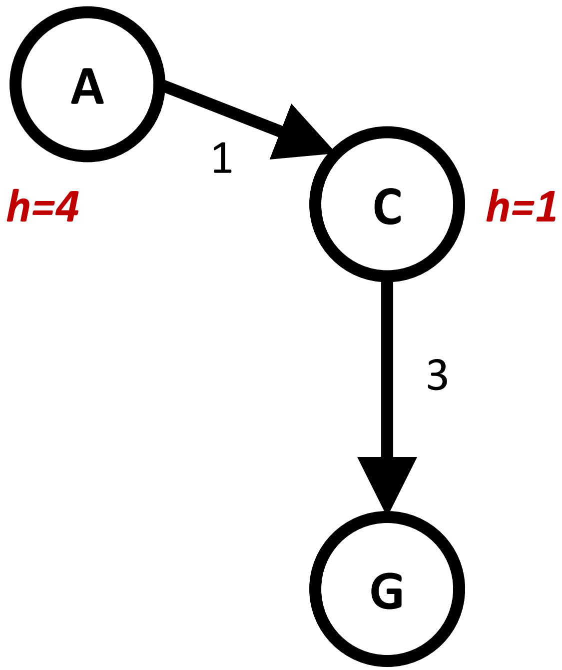
&\dots \\
&\leq \textstyle\sum_{i=1}^{k} c(n_i, a_i, n_{i+1}) + h(G) \; \mathbf{0}
\end{aligned}
$$

$\Rightarrow h(n_1) \leq$ cost of (every) path from n_1 to goal
$\leq$ cost of optimal path from n_1 to goal

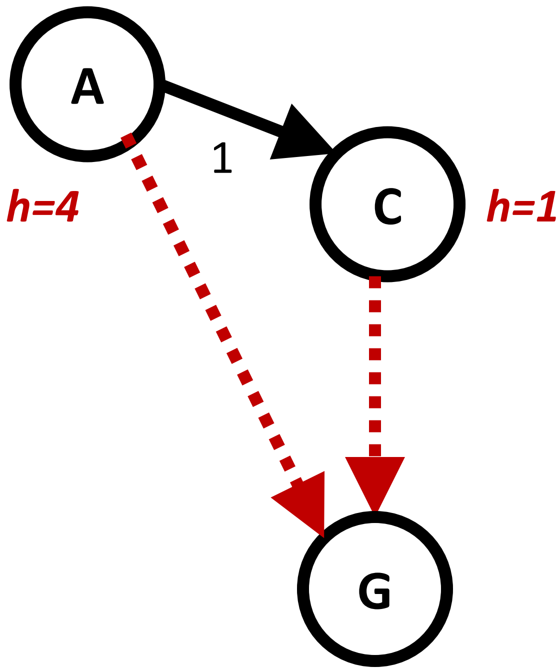
Consistency of Heuristics



- Main idea: estimated heuristic costs ≤ actual costs
  - Admissibility: heuristic cost ≤ actual cost to goal

    h(A) ≤ actual cost from A to G

Consistency of Heuristics



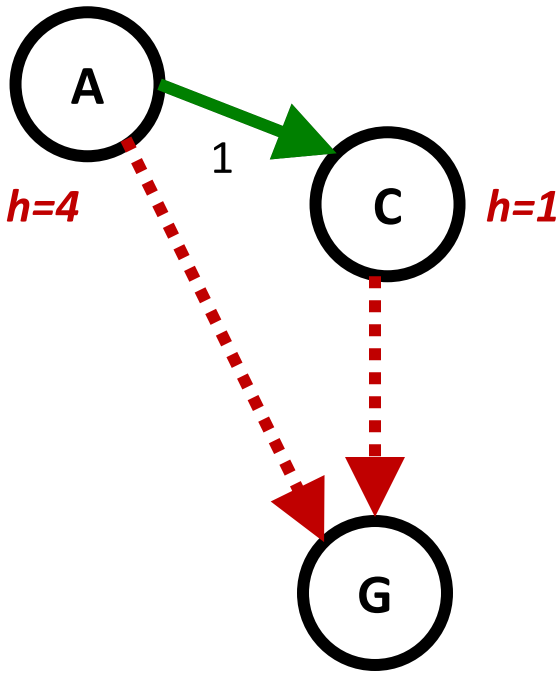
- Main idea: estimated heuristic costs ≤ actual costs
  - Admissibility: heuristic cost ≤ actual cost to goal

    h(A) ≤ actual cost from A to G

  - Consistency: heuristic “arc” cost ≤ actual cost for each arc

    h(A) – h(C) ≤ cost(A to C)

Consistency of Heuristics



- Main idea: estimated heuristic costs ≤ actual costs
  - Admissibility: heuristic cost ≤ actual cost to goal

    h(A) ≤ actual cost from A to G

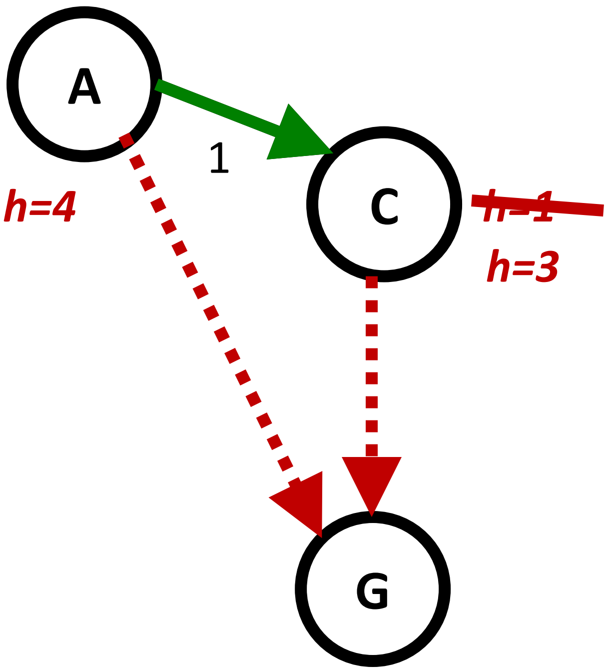
  - Consistency: heuristic “arc” cost ≤ actual cost for each arc

    h(A) – h(C) ≤ cost(A to C)

    or h(A) ≤ c(A,C) + h(C) (triangle inequality)

    Note: h* *necessarily* satisfies triangle inequality

Consistency of Heuristics



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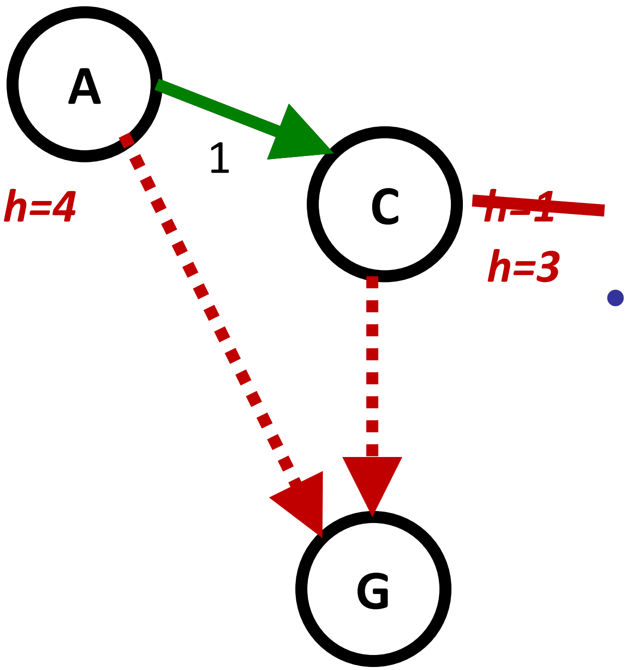
Consistency of Heuristics



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  - Admissibility: heuristic cost ≤ actual cost to goal

    h(A) ≤ actual cost from A to G

  - Consistency: heuristic "arc" cost ≤ actual cost for each arc

    h(A) – h(C) ≤ cost(A to C)

- Consequences of consistency:
  - The f value along a path never decreases

    h(A) ≤ cost(A to C) + h(C)

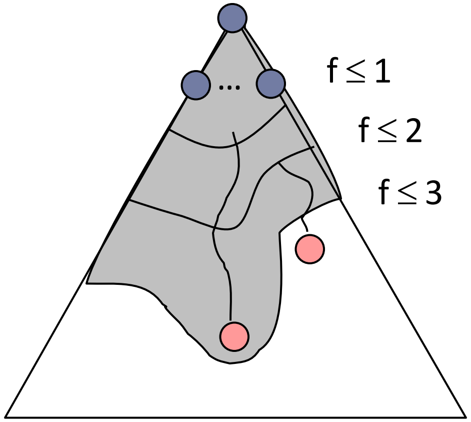
    => g(A) + h(A) ≤ g(A) + c(A,C) + h(C)

    => f(A) ≤ g(C)+h(C)=f(C)

  - A* graph search is optimal

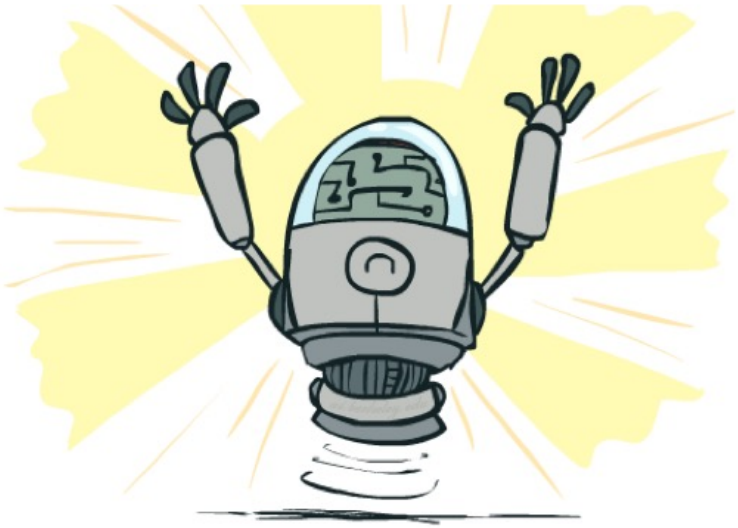
Optimality of A* Graph Search

- Sketch: consider what A* does with a consistent heuristic:
  - Fact 1: A* expands nodes in increasing total f value (f-contours)
  - Fact 2: For every state s, nodes that reach s optimally are expanded before nodes that reach s suboptimally
  - Result: A* graph search is optimal



Optimality

- Tree search:
  - A* is optimal if heuristic is admissible
  - UCS is a special case (h = 0)



- Graph search:
  - A* optimal if heuristic is consistent
  - UCS optimal (h = 0 is consistent)
- Consistency implies admissibility
- In general, most natural admissible heuristics tend to be consistent, especially if from relaxed problems

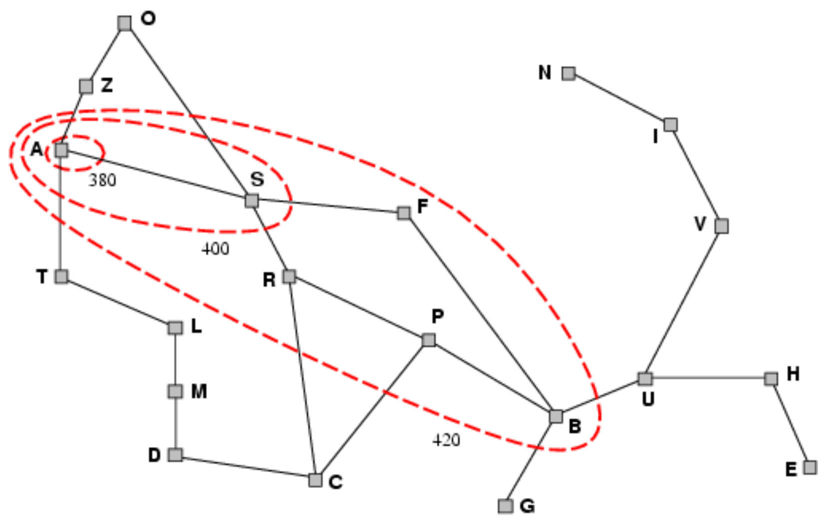
Admissible vs. Consistent (Tree vs. Graph Search)

- Consistent heuristic: When selecting a node for expansion, the path with the lowest cost to that node has been found

- When an admissible heuristic is not consistent, a node will need repeated expansion, every time a new best (so-far) cost is achieved for it.

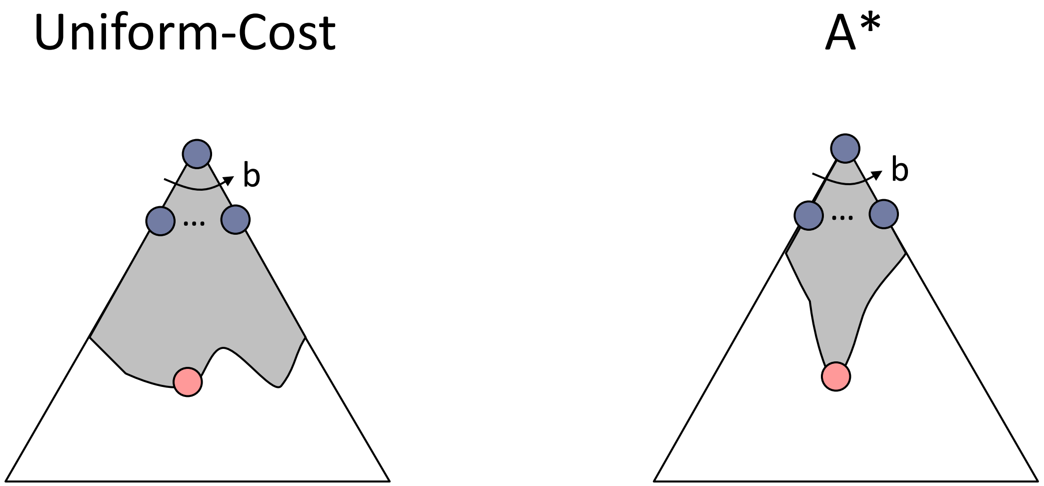
Contours in the state space

- A* (using `GRAPH-SEARCH`) expands nodes in order of increasing f value
- Gradually adds "f-contours" of nodes
  - Contour i has all nodes with $f = f_i$ where $f_i < f_{i+1}$



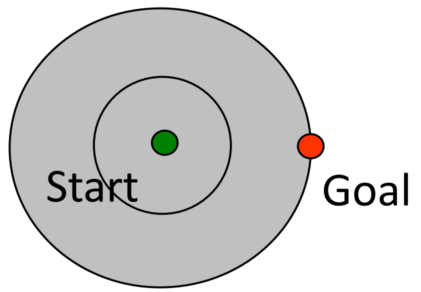
A* expands all nodes with f(n) < C*
A* expands some nodes with f(n) = C* (nodes on the goal contour)
A* expands no nodes with f(n) > C* ⟹ pruning

Properties of A*

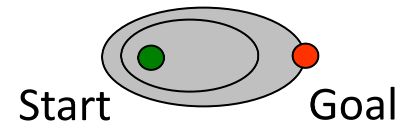


UCS vs A* Contours

- Uniform-cost (A* using $h(n)$ = 0) expands equally in all "directions"



- A* expands mainly toward the goal, but does hedge its bets to ensure optimality
  - More accurate heuristics stretched toward the goal (more narrowly focused around the optimal path)



States are points in 2-D Euclidean space.
g(n)=distance from start
h(n)=estimate of distance from goal

Video of Demo Contours (Empty) -- UCS

Video of Demo Contours (Empty) -- Greedy

Video of Demo Contours (Empty) – A*

Video of Demo Contours (Pacman Small Maze) – A*

Comparison

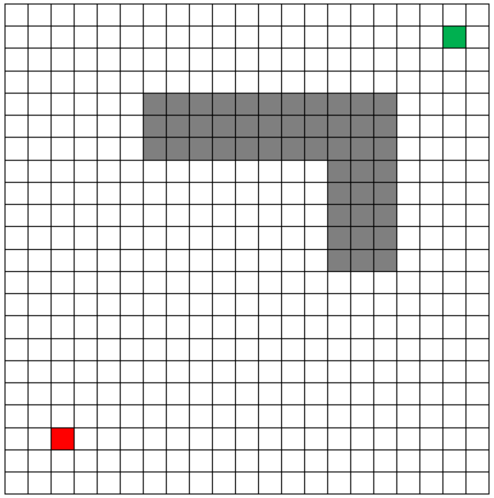
Greedy (h)

Uniform Cost (g)

A* (g+h)

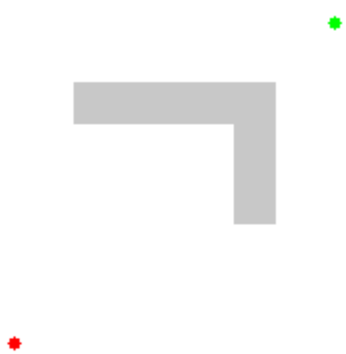
Robot navigation example

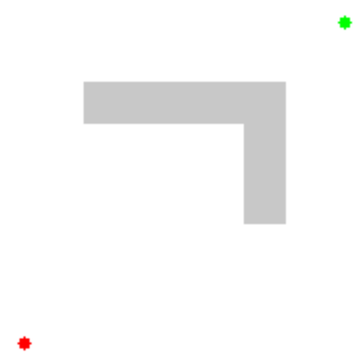
- Initial state? Red cell
- States? Cells on rectangular grid (except to obstacle)
- Actions? Move to one of 8 neighbors (if it is not obstacle)
- Goal test? Green cell
- Path cost? Action cost is the Euclidean length of movement



A* vs. UCS: Robot navigation example

- Heuristic: Euclidean distance to goal
- Expanded nodes: filled circles in red & green
  - Color indicating g value (red: lower, green: higher)
- Frontier: empty nodes with blue boundary
- Nodes falling inside the obstacle are discarded



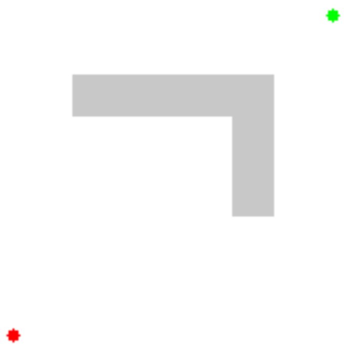


Adopted from: http://en.wikipedia.org/wiki/A*_search_algorithm

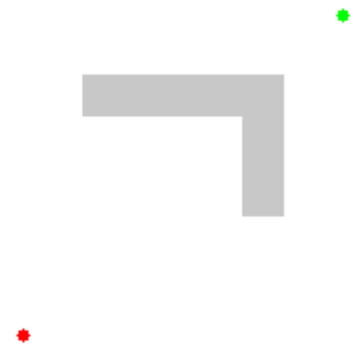
Robot navigation: Admissible heuristic

- Is Manhattan $d_M(x, y) = |x_1 - y_1| + |x_2 - y_2|$ distance an admissible heuristic for previous example?

A*: Inadmissible heuristic



$h = 5 * h\_SLD$



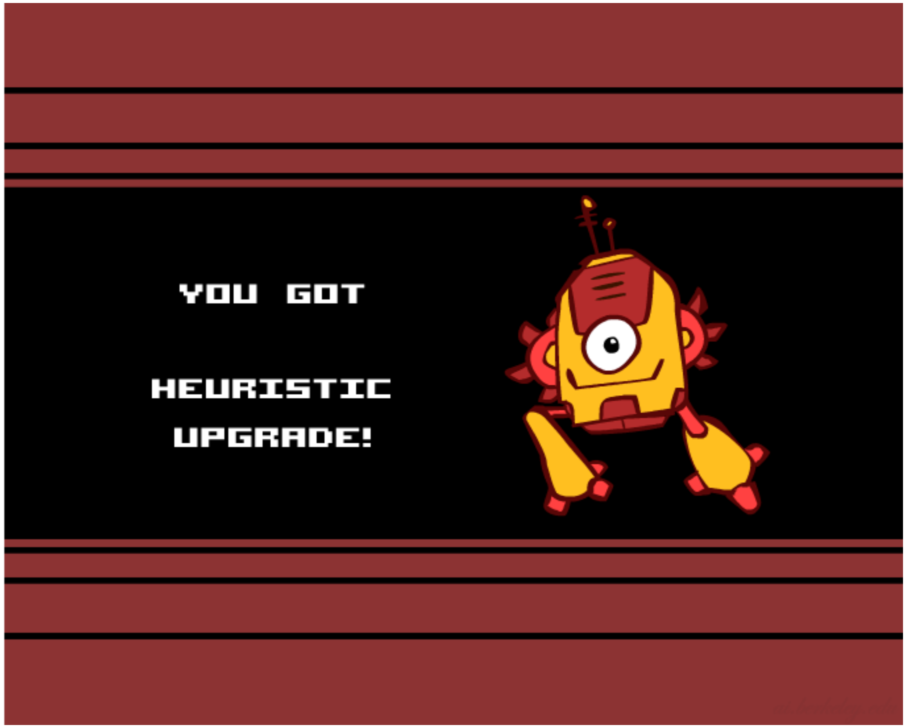
$h = h\_SLD$

Adopted from: http://en.wikipedia.org/wiki/A*_search_algorithm

A*: Summary

- A* orders nodes in the queue by $f(n) = g(n) + h(n)$
  - A* uses both backward costs and (estimates of) forward costs
- A* is optimal for trees/graphs with admissible/consistent heuristics
- Heuristic design is key: often use relaxed problems

Creating Heuristics



Relaxed problem

- Relaxed problem: Problem with fewer restrictions on the actions

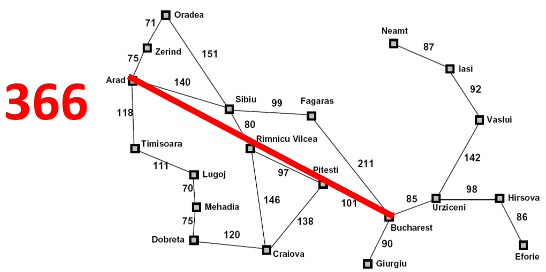
- Optimal solution to the relaxed problem may be computed easily (without search)

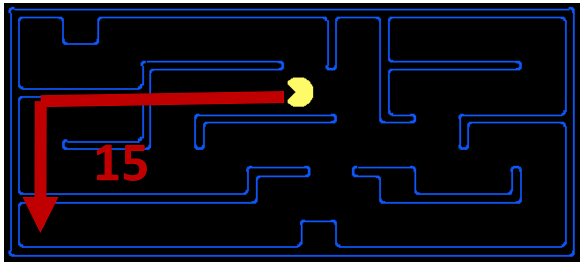
- The cost of an optimal solution to a relaxed problem is an admissible heuristic for the original problem
  - The optimal solution is the shortest path in the super-graph of the state-space.

Creating Admissible Heuristics

- Most of the work in solving hard search problems optimally is in coming up with admissible heuristics

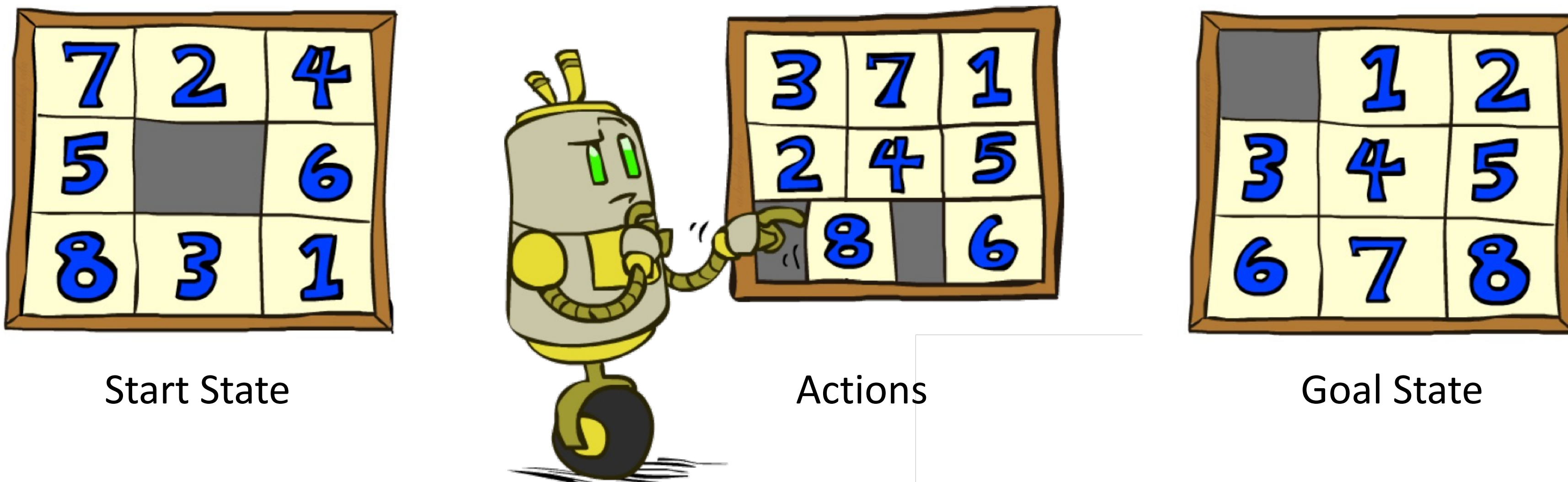
- Often, admissible heuristics are solutions to *relaxed problems,* where new actions are available





- Problem P_2 is a relaxed version of P_1 if $\mathcal{A}_2(s) \supseteq \mathcal{A}_1(s)$ for every s
- Theorem: $h_2^*(s) \leq h_1^*(s)$ for every s, so $h_2^*(s)$ is admissible for P_1
- Inadmissible heuristics are often useful too

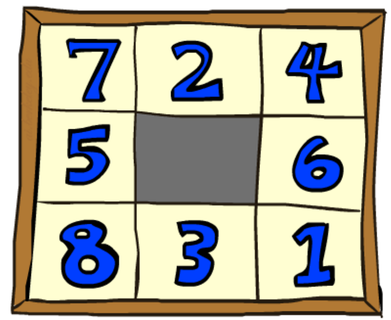
Example: 8 Puzzle



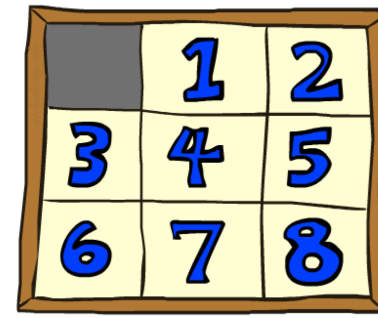
- What are the states?
- How many states?
- What are the actions?
- What are the step costs?

8 Puzzle I

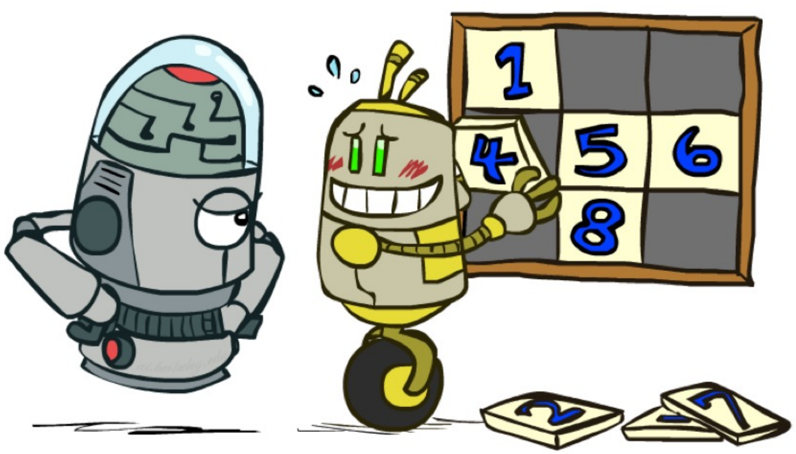
- Heuristic: Number of tiles misplaced
- Why is it admissible?
- h(start) = 8



Start State



Goal State

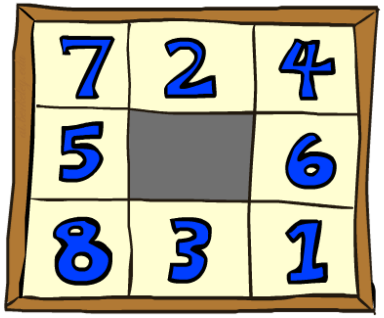


	Average nodes expanded when the optimal path has...		
	...4 steps	...8 steps	...12 steps
UCS	112	6,300	3.6×10^6
A*TILES	13	39	227

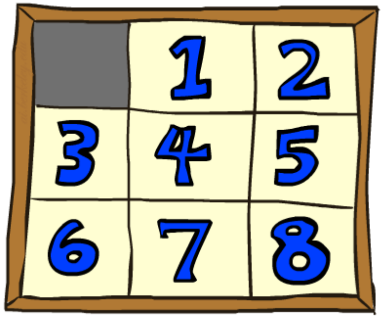
Statistics from Andrew Moore

8 Puzzle II

- What if we had an easier 8-puzzle where any tile could slide any direction at any time, ignoring other tiles?
- Total *Manhattan* distance= sum of Manhattan distance of tiles from their target position
- Why is it admissible?
- h(start) = 3 + 1 + 2 + … = 18



Start State



Goal State

	Average nodes expanded when the optimal path has…		
	…4 steps	…8 steps	…12 steps
A*TILES	13	39	227
A*MANHATTAN	12	25	73

Relaxed problem: 8-puzzle

- 8-Puzzle: move a tile from square A to B if A is adjacent (left, right, above, below) to B and B is blank
  - Relaxed problems
    1. can move from A to B if A is adjacent to B (ignore whether or not position is blank)
    2. can move from A to B if B is blank (ignore adjacency)
    3. can move from A to B (ignore both conditions)
- Admissible heuristics for original problem ($h_1(n)$ and $h_2(n)$) are optimal path costs for relaxed problems
  - First case: a tile can move to any adjacent square $\Rightarrow h_2(n)$
  - Third case: a tile can move anywhere $\Rightarrow h_1(n)$

Combining heuristics

- Dominance: $h_1 \geq h_2$ if $\forall n: h_1(n) \geq h_2(n)$
  - Roughly speaking, larger is better as long as both are admissible
  - The zero heuristic is pretty bad (what does A* do with h=0?)
  - The exact heuristic is pretty good, but usually too expensive!

- What if we have two heuristics, neither dominates the other?
  - Form a new heuristic by taking the max of both:

    $$h(n) = \max(h_1(n), h_2(n))$$

  - Max of admissible heuristics is admissible and dominates both!

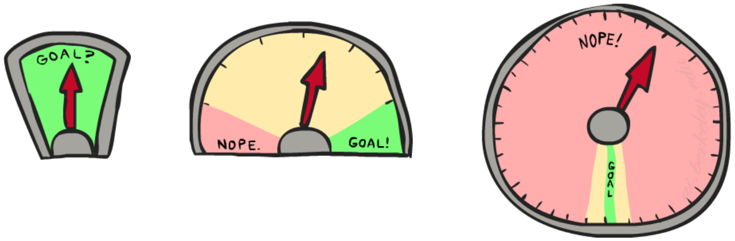
Heuristic quality

- If $\forall n, h_2(n) \geq h_1(n)$ (both admissible)
 then h_2 dominates h_1 and it is better for search

- Surely expanded nodes: $f(n) < C^* \Rightarrow h(n) < C^* - g(n)$
  - If $h_2(n) \geq h_1(n)$ then every node expanded for h_2 will also be surely expanded with h_1 (h_1 may also causes some more node expansion)

8 Puzzle: heuristic

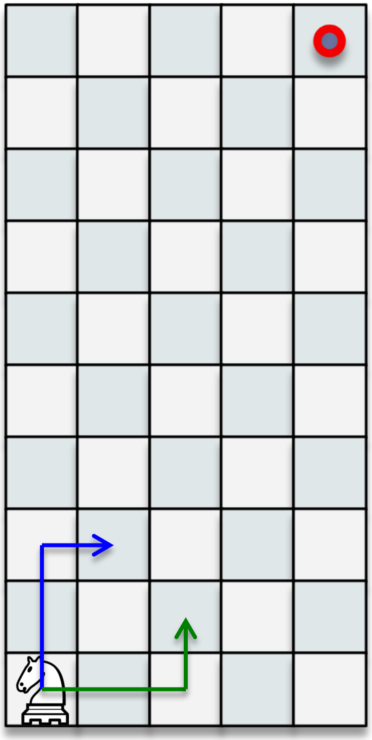
- How about using the *actual cost* as a heuristic?
  - Would it be admissible?
  - Would we save on nodes expanded?
  - What's wrong with it?



- With A*: a trade-off between quality of estimate and work per node
  - As heuristics get closer to the true cost, you will expand fewer nodes but usually do more work per node to compute the heuristic itself

Example: Knight's moves

- Minimum number of knight's moves to get from A to B?
  - h_1 = (Manhattan distance)/3
    - h_1' = h_1 rounded up to correct parity (even if A, B same color, odd otherwise)
  - h_2 = (Euclidean distance)/√5 (rounded up to correct parity)
  - h_3 = (max x or y shift)/2 (rounded up to correct parity)
- $h(n) = \max(h_1'(n), h_2(n), h_3(n))$ is admissible!



A* Applications

- Video games
- Pathing / routing problems
- Resource planning problems
- Robot motion planning
- Language analysis
- Machine translation
- Speech recognition
- Protein design
- Chemical synthesis
- …

Memory bounded methods

- A* keeps the entire explored region in memory
- => will run out of space before you get bored waiting for the answer
- There are variants that use less memory (Section 3.5.5):
  - IDA* works like IDS, except it uses an *f*-limit instead of a depth limit
  - RBFS is a recursive depth-first search that uses an *f*-limit = the *f*-value of the best alternative path available from any ancestor of the current node
  - SMA* uses *all available memory* for the queue, minimizing thrashing